GONALITY AND CLIFFORD INDEX OF CURVES ON ELLIPTIC K3 SURFACES WITH PICARD NUMBER TWO

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Abstract. We compute the Clifford index of all curves on K3 surfaces with Picard group isomorphic to $U(m)$.

1. Introduction

In the past years many authors have studied problems related to the gonality and Clifford index of curves lying on K3 surfaces. In this paper, by *curve* we always mean a smooth, reduced and irreducible curve over the field of complex numbers. The gonality and the Clifford index of a curve C are respectively defined by

$$\mathrm{gon}(C) = \min\{\deg(A) : A \in \mathrm{Div}(C),\ h^0(A) = 2\},$$
$$\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(A) : A \in \mathrm{Div}(C),\ h^0(A) \geq 2,\ h^1(A) \geq 2\}$$

where $\mathrm{Cliff}(A) = \deg A - 2h^0(A) + 2$. The Clifford index measures how special C is in the moduli space $\mathcal{M}_g$ of curves of genus g, in the following sense. One has

$$0 \leq \mathrm{Cliff}(C) \leq \lfloor \tfrac{g-1}{2} \rfloor,$$

where the second inequality is an equality for the generic member of $\mathcal{M}_g$, and on the other hand $\mathrm{Cliff}(C) = 0$ if and only if C is hyperelliptic (cf. [1]). Therefore, in this paper we say that a curve C is *Clifford general* if $\mathrm{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$. In all other cases, we say that C is *Clifford special*.

A classical result by Saint-Donat [14, 5.8] states that given a hyperelliptic curve C on a K3 surface, all the curves in the linear system $|C|$ are also hyperelliptic. This interesting fact was vastly generalized by Green and Lazarsfeld [9], who proved that indeed the Clifford index is the same for each smooth member of $|C|$. They also proved that, whenever C is Clifford special, there exists a divisor D on the ambient K3 surface X whose restriction to C computes its Clifford index[1]. In this case one says that *the Clifford index is cut out on C by D*. In [11] Knutsen showed that, if C is Clifford special, we can choose D to be a (smooth and irreducible) curve, and moreover $\mathrm{Cliff}(C) = D.(C - D) - 2$.

Originally, the constancy of the gonality of curves in a linear system on a K3 surface had been conjectured (unpublished) by Harris and Mumford, but Donagi and Morrison [7] found a counterexample. However, by the works of Ciliberto and Pareschi [5] and Knutsen [12] we know that this is indeed the only counterexample.

On the other hand, the notions of gonality and Clifford index are very much related: for any curve C of Clifford index c one has

$$c + 2 \leq \mathrm{gon}(C) \leq c + 3,$$

[1]A divisor $A \in \mathrm{Div}(C)$ is said to compute the Clifford index of C if it appears in the definition of $\mathrm{Cliff}(C)$ and achieves the minimal value, i.e. $\mathrm{Cliff}(A) = \mathrm{Cliff}(C)$.

and curves for which $\mathrm{gon}(C) = c+3$ are conjectured to be very rare (cf. [8]). When lying on K3 surfaces, these curves are completely classified by Knutsen [12].

In general, the gonality and the Clifford index are subtle invariants which are hard to compute explicitly for a given curve. In this note, we compute the Clifford index and gonality of all curves on some elliptic K3 surfaces. We prove the following.

Theorem 1.1. *Let X be a K3 surface with Picard group isomorphic to $U(m)$, with $m \in \mathbb{Z}$, $m \geq 1$. Denote by E and F two generators of $\mathrm{Pic}(X)$, with $E^2 = F^2 = 0$ and $E.F = m$. Let C be a curve on X of genus $g > 2$. Then, either*

- *(i) The Clifford index of C is cut out on C by an elliptic curve E_C, which is linearly equivalent to the one among E and F having minimal intersection with C. Then $\mathrm{Cliff}(C) = C.E_C - 2$ and $C.E_C$ is equal to the gonality of C; or*
- *(ii) $m > 2$ and C is linearly equivalent to $E + F$. Then C has maximal Clifford index $\mathrm{Cliff}(C) = \lfloor m/2 \rfloor$.*

In the statement of the theorem U denotes the *hyperbolic lattice*: the lattice given by $\mathbb{Z} \oplus \mathbb{Z}$ with intersection matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$U(m)$ denotes the lattice obtained by U by multiplying the intersection matrix by a non-zero integer m. The term *isomorphic* means isomorphic as lattices.

Notice that there always exists a class of square zero in $\mathrm{Pic}(X) \simeq U(m)$. Therefore a K3 surface X as in the theorem admits an elliptic fibration by [13, §3].

Part of the motivation for studying this problem came from a recent paper by Watanabe [16] in which he shows that for a K3 surface X which is a double cover of a smooth del Pezzo surface of degree $4 \leq d \leq 8$ such that X carries a non-symplectic automorphism of order two which acts trivially on the Picard group, then for any curve C on X, either the Clifford index is cut out on C by some elliptic curve on X, or C is linearly equivalent to a multiple of a curve of genus 2. The key idea of Watanabe is that the automorphism yields some useful geometric informations which help to characterize the topological properties of the curves on X.

In this work, we started to investigate the analogue situation when X carries a non-symplectic automorphism of order 3 which acts trivially on the Picard group. For $\rho(X) = 2$, we know by the classification results of Artebani-Sarti [2] and Taki [15], that the Picard group of X is isomorphic to either U or $U(3)$. Therefore, our result applies to this case. However, not all K3 surfaces with Picard group $U(m)$ admit non-symplectic automorphisms: see [2], and also Artebani-Sarti-Taki [3].

Notation and conventions. We work over the complex number field $\mathbb{C}$. By *surface* we mean a smooth irreducible projective surface. A K3 surface is a regular surface with trivial canonical bundle. The Picard number of a surface X is by definition the rank of the Picard group $\mathrm{Pic}(X)$ and is denoted by the letter ρ. The symbol $\sim$ denotes linear equivalence between divisors and $|D|$ is used to denote the complete linear system associated to a divisor D. A lattice is a free $\mathbb{Z}$-module L of finite rank equipped with a non-degenerate symmetric integral bilinear form $L \times L \to \mathbb{Z}$, $(x, y) \mapsto x.y$. An isomorphism of lattices is a $\mathbb{Z}$-module isomorphism preserving the bilinear forms.

2. Clifford special curves on a K3 surface

Let X be a K3 surface. In this short section we recall some fundamental results which will be needed in the following and we also explain why the case $\rho(X) = 1$ is not interesting for our purposes. Fix a curve C of genus g on X. Let

$$\mathcal{A}(C) := \{D \in \mathrm{Div}(X) \mid h^0(\mathcal{O}_X(D)) \geq 2,\ h^0(\mathcal{O}_X(C - D)) \geq 2\}.$$

Notice that C admits a decomposition $C \sim D + D'$ into two moving classes D and D' if and only if $\mathcal{A}(C) \neq \emptyset$. When this happens, among such decompositions it is interesting to consider those with minimal intersection $D.D'$. Hence, one defines

$$\mu_C := \min\{D.(C - D) - 2 \mid D \in \mathcal{A}(C)\},$$

and denotes by $\mathcal{A}^0(C)$ the divisors in $\mathcal{A}(C)$ achieving this minimal value:

$$\mathcal{A}^0(C) := \{D \in \mathcal{A}(C) \mid D.(C - D) - 2 = \mu_C\} \subset \mathcal{A}(C).$$

Observe that $\mu_C \geq 0$ since the curve C (or any member of the complete linear system of a basepoint free and big line bundle on a K3 surface) is numerically 2-connected (cf. [14, (3.9.6)]). By the results in [9] and [11] we have: (cf. [10, p.11])

$$\mathrm{Cliff}(C) = \min\{\mu_C, \lfloor \tfrac{g-1}{2} \rfloor\}.$$

In other words, either C is Clifford general, or C is Clifford special and then $\mathcal{A}(C)$ is non-empty, the Clifford index of C is cut out by some divisor $D \in \mathcal{A}^0(C)$ on X and $\mathrm{Cliff}(C) = \mu_C$. Then, by definition of Clifford index

$$h^0(\mathcal{O}_C(D)) \geq 2 \ \text{ and } \ h^1(\mathcal{O}_C(D)) = h^0(\omega_C \otimes \mathcal{O}_C(-D)) \geq 2.$$

In particular, the linear systems on the curve C given by the line bundles $\mathcal{O}_C(D)$ and $\omega_C \otimes \mathcal{O}_C(-D) = \mathcal{O}_C(C - D)$ contain some non-trivial effective divisors, hence of positive degree. Therefore we get $\deg_C(C - D) = C.(C - D) > 0$ and also $\deg_C(D) = C.D > 0$. Altogether, this yields the following inequalities:

$$0 < C.D < C^2.$$

In particular, when $\mathrm{Pic}(X) = \mathbb{Z}[H]$, we see that this inequalities are impossible when $C \in |H|$, and so C has general Clifford index in this case. On the other hand, for $C \sim kH$ with $k \geq 2$, a direct computation shows that the Clifford index of C is cut out on C by a member of $|H|$. When $\rho(X) \geq 2$, however, the situation is more interesting. In the next section we compute the Clifford index of any curve on a K3 surface with Picard group isomorphic to $U(m)$.

3. Proof of the Theorem

Let X be a K3 surface with Picard group isomorphic to $U(m)$, with $m \geq 1$.

We let the Picard group of X be generated by the classes of two effective divisors E and F such that $E^2 = F^2 = 0$ and $E.F = m$. Up to the action of the Weyl group of X we may assume that E is an elliptic curve (cf. [13, §3]).

When $m = 1$, we observe that the rational curve $\Gamma \sim F - E$ yields a section of the elliptic fibration given by $|E|$. Moreover, the linear system $|F| = |E + \Gamma|$ contains a rational curve as a base component and therefore F cannot be represented by an irreducible curve (cf. Saint-Donat [14, 2.6 & 2.7]).

On the other hand, when $m > 1$, since $x^2 \in 2m\mathbb{Z}$ for $x \in U(m)$ we observe that there are no rational curves on X. Thus any effective divisor is nef and basepoint

free (ibid.). In particular we may assume that F is an elliptic curve. A simple computation shows that any elliptic curve on X belongs to either $|E|$ or $|F|$.

For any effective divisor C on X let us define

$$\begin{array}{lll} d_C & := & \min\{E' \cdot C \mid E' \text{ is an elliptic curve on } X\}, \\ \mathcal{E}^0(C) & := & \{\text{elliptic curves } E_C \text{ such that } E_C \cdot C = d_C\}. \end{array}$$

Lemma 3.1. *Let C be a curve with $C^2 > 0$ and let E_C be an elliptic curve in $\mathcal{E}^0(C)$. If $(C - E_C)^2 = 0$, then C belongs to the linear system $|E + F|$.*

Proof. If $(C - E_C)^2 = 0$ then $C - E_C$ is linearly equivalent to a multiple of an elliptic curve E', so that we can write $C = E_C + (C - E_C) \sim E_C + kE'$, some $k \geq 1$. Since $C^2 > 0$ we see that E' is not linearly equivalent to E_C. Since $E'.C = E_C.C$, we get $k = 1$ and $C \sim E_C + E' \sim E + F$. □

Lemma 3.2. *Let $m \geq 2$ and let C be a curve in the linear system $|E + F|$. Then*

(i) If $m = 2$ then C is Clifford special.
(ii) If $m > 2$ then C is Clifford general.

Proof. Assume $D \in \mathcal{A}^0(C)$ and let $D \sim aE + bF$, with $a, b \geq 0$. Then by definition of $\mathcal{A}(C)$ we may assume $C - D$ effective, so that $0 \leq (C - D).E = m(1 - b)$ and $0 \leq (C - D).F = m(1 - a)$. Hence $a, b \in \{0, 1\}$. This shows that the only curves D in $\mathcal{A}^0(C)$ are the members of $|E|$ and $|F|$. Then C is Clifford special whenever $\mu_C = D.(C - D) - 2 = m - 2 < \lfloor C^2/4 \rfloor = \lfloor m/2 \rfloor$, that is for $m \leq 2$. □

Remark 3.3. The case $m = 1$ is not to be considered here since the linear system $|E + F|$ contains a rational curve $\Gamma \sim F - E$ as base component; hence there are no (irreducible) curves in $|E + F|$ in this case.

Lemma 3.4. *Let $C \subset X$ be an effective divisor with $C^2 > 0$. For any elliptic curve E' on X we have*

$$(C - E')^2 \geq 0, \quad h^0(C - E') \geq 2.$$

Moreover, $|C - E'|$ is basepoint free for $m > 1$.

Proof. Since E and F are the only effective reduced divisors with self-intersection zero, it is clear that in order to show the Lemma we may assume $E' \in |E|$, by the symmetry of the roles of E and F in $\mathrm{Pic}(X)$. Let $C \sim aE + bF$ for some positive integers a and b. Then clearly $(C - E)^2 \geq 0$ and also $C.E > 0$. Thus $E.(C - E) > 0$, which shows that $C - E$ is effective. It follows $h^0(C - E) \geq 2$ by Riemann-Roch. Moreover, if $m > 1$, then $|C - E|$ is basepoint free since in this case there are no rational curves on X (cf. [14, §2.7]). □

Remark 3.5. Let C be a curve on X. By the definition of $\mathcal{A}(C)$ and Lemma 3.4 above $\mathcal{E}^0(C) \subset \mathcal{A}(C)$. In particular $\mu_C \leq d_C - 2$. Moreover,

$$\mathcal{E}^0(C) \subset \mathcal{A}^0(C) \iff \mu_C = d_C - 2.$$

Indeed, let $E_C \in \mathcal{E}^0(C) \subset \mathcal{A}(C)$. If $\mu_C = d_C - 2$ then E_C computes μ_C. Hence $E_C \in \mathcal{A}^0(C)$. The other implication is obvious.

Proof of Theorem 1.1. The first (and longer) part of the proof is to show that

$$\mathcal{E}^0(C) \subset \mathcal{A}^0(C).$$

Let $E_C \in \mathcal{E}^0(C)$. By Lemma 3.4, $E_C \in \mathcal{A}(C)$, so that $\mathcal{A}^0(C)$ is not empty.

If $\mathcal{A}^0(C)$ contains some elliptic curve F, then $C.E_C \leq C.F$ and so $C.E_C - 2 \leq \mu_C$. Since $E_C \in \mathcal{A}(C)$, we have $C.E_C - 2 = \mu_C$. Therefore $E_C \in \mathcal{A}^0(C)$.

So we assume that $\mathcal{A}^0(C)$ contains no elliptic curves at all. Let D be an effective divisor in $\mathcal{A}^0(C)$. Since $D \in \mathcal{A}(C)$, and $h^1(D) = 0$ by [10, Prop. 2.6], we have $D^2 \geq 0$. Let us show that, in fact,

$$D^2 \geq 2.$$

Indeed, assume by contradiction $D^2 = 0$.

- In the case where $m \geq 2$, since X contains no rational curves, D is basepoint free, and so it is linearly equivalent to an elliptic curve, by [14, 2.6]. This contradicts the assumption that $\mathcal{A}^0(C)$ contains no elliptic curves.
- In the case where $m = 1$, let E and F be generators of the Picard group of X, with $E^2 = F^2 = 0$ and $E.F = 1$. Then we may assume that E is an elliptic curve and there exists a rational curve Γ on X such that $\Gamma \sim F - E$. Since $D^2 = 0$ and $D \in \mathcal{A}^0(C)$, by [10, Prop. 2.6] we have $D \sim E$ or F. However, E is not in $\mathcal{A}^0(C)$ by assumption, thus $D \sim F$. Since Γ is the base locus of F, we have $C.(F - E) = 0$. On the other hand,
$$\mu_C = C \cdot D - 2 = C \cdot E - 2$$
and, since $E \in \mathcal{A}(C)$, this yields $E \in \mathcal{A}^0(C)$. A contradiction.

By the above discussion we always have $D^2 \geq 2$. Notice that $C - D \in \mathcal{A}^0(C)$ and then $(C - D)^2 \geq 2$ by the same reason. We want to show that $E_C \in \mathcal{A}^0(C)$, contradicting the assumption that $\mathcal{A}^0(C)$ contains no elliptic curves. Concretely, we need to show the following inequality

$$E_C \cdot C \leq D \cdot (C - D).$$

Rewrite this inequality as

$$(D - E_C) \cdot (D' - E_C) \geq 0, \quad D' := C - D \tag{3.1}$$

For $E_D \in \mathcal{E}^0(D)$ and $E_{D'} \in \mathcal{E}^0(D')$ we let

$$\begin{aligned} n_D &= (D - E_D).(D' - E_{D'}) \\ r_D &= D.(E_{D'} - E_C) \\ r_{D'} &= D'.(E_D - E_C) \end{aligned}$$

so that we may now rewrite (3.1) as follows:

$$n_D + r_D + r_{D'} \geq E_D \cdot E_{D'} \tag{3.2}$$

Claim. For *any choice* of elliptic curves $E_D \in \mathcal{E}^0(D)$ and $E_{D'} \in \mathcal{E}^0(D')$,

$$n_D = (D - E_D).(D' - E_{D'}) \geq 0.$$

Indeed, by Lemma 3.4 the classes of $(D - E_D)$ and $(D' - E_{D'})$ have non-negative self-intersection and are effective, thus they lie in the closure of the positive cone and intersect non-negatively (cf. [4, IV.7]). This proves our claim.

Now, consider the following inequalities:

$$\begin{aligned} r_D &\geq r_D + C.(E_C - E_{D'}) = D'.(E_C - E_{D'}) \geq 0 \\ r_{D'} &\geq r_{D'} + C.(E_C - E_D) = D.(E_C - E_D) \geq 0 \end{aligned} \tag{3.3}$$

If we assume that either $r_D > 0$ or $r_{D'} > 0$ then (3.2) holds, since $n_D \geq 0$ and

$$E_C \cdot E_{D'} \leq m \quad \text{and} \quad r_D \geq m \quad \text{or} \quad r_{D'} \geq m.$$

(recall that $x.y \in m\mathbb{Z}$ for $x, y \in U(m)$). Hence, we assume $r_D = 0$ and $r_{D'} = 0$. Substituting this in (3.3) we get $D.E_C = D.E_D$ and $D'.E_C = D'E_{D'}$. Thus,

$$E_C \in \mathcal{E}^0(D) \cap \mathcal{E}^0(D').$$

Using the claim above, we can replace both E_D and $E_{D'}$ by E_C in the definition of n_D and this yields the desired inequality (3.1) and therefore $\mathcal{E}^0(C) \subset \mathcal{A}^0(C)$.

Now that we know $\mathcal{E}^0(C) \subset \mathcal{A}^0(C)$, we determine all Clifford general curves. Take $E_C \in \mathcal{E}^0(C)$. By Lemma 3.4 we know $C - E_C \in \mathcal{A}(C)$. Moreover, we also have $C - E_C \in \mathcal{A}^0(C)$ since $E_C \in \mathcal{A}^0(C)$ by assumption. We distinguish two cases:

- $(C - E_C)^2 = 0$. Then, by Lemma 3.1 and Lemma 3.2, C is Clifford general if and only if $m > 2$ and $C \in |E + F|$.
- $(C - E_C)^2 > 0$. (in particular C is not linearly equivalent to $E + F$). We then show that C is Clifford special. This amounts to show

$$2\mu_C \leq g - 3$$

which, by the definition of μ_C and the genus formula, is equivalent to

$$(C - 2E_C)^2 \geq -4.$$

We may write $C \sim aE_C + D$, with $a \geq 1$, D effective and $D^2 = 0$. If $a = 1$ we get $C \in |E + F|$ by Lemma 3.1, which is not the case. So $a \geq 2$ and

$$(C - 2E_C)^2 \geq 0.$$

Therefore, C is Clifford special.

This proves that C is Clifford general if and only if $m > 2$ and $C \in |E + F|$, as in part (ii) of the Theorem. To show part (i), we can therefore assume that C is Clifford special. Then $\mathrm{Cliff}(C) = \mu_C$ and since $\mathcal{E}^0(C) \subset \mathcal{A}^0(C)$ we have $\mu_C = d_C - 2$. Therefore, the Clifford index of C is cut out by some elliptic curve $E_C \in \mathcal{E}^0(C)$. In particular, $\mathrm{Cliff}(C)$ is computed by a pencil: the restriction of $|E_C|$ to C. Therefore (cf. [8, p.174])

$$\mathrm{gon}(C) = \mathrm{Cliff}(C) + 2 = d_C.$$

Hence, the assertions of (i) follow and the Theorem is proved. □

Remark 3.6. In particular, we observe that when $m = 1$ or 2, any curve on X is Clifford special and its Clifford index is cut out by an elliptic curve. The same conclusion when $m = 2$ is implicitly contained in [16].

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References

[1] E. Arbarello, M. Cornalba, P. Griffiths, and J.D. Harris. *Geometry of Algebraic Curves*. Number v. 1 in Grundlehren der mathematischen Wissenschaften. Springer New York, 2010.

[2] M. Artebani and A. Sarti. Non-symplectic automorphisms of order 3 on K3 surfaces. *Mathematische Annalen*, 342(4):903–921, 2008.

[3] M. Artebani, A. Sarti, and S. Taki. K3 surfaces with non-symplectic automorphisms of prime order. *Mathematische Zeitschrift*, 268(1-2):507–533, 2011.

[4] W. Barth, K. Hulek, C. Peters, and A. van de Ven. *Compact Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2014.

[5] Ciro Ciliberto and Giuseppe Pareschi. Pencils of minimal degree on curves on a $K3$ surface. *J. Reine Angew. Math.*, 460:15–36, 1995.

[6] I.V. Dolgachev and S. Kondo. Moduli of K3 Surfaces and Complex Ball Quotients. In Rolf-Peter Holzapfel, A.Muhammed Uluda, and Masaaki Yoshida, editors, *Arithmetic and Geometry Around Hypergeometric Functions*, volume 260 of *Progress in Mathematics*, pages 43–100. Birkhuser Basel, 2007.

[7] Ron Donagi and David R. Morrison. Linear systems on $K3$-sections. *J. Differential Geom.*, 29(1):49–64, 1989.

[8] D. Eisenbud, H. Lange, G. Martens, and F. Schreyer. The Clifford dimension of a projective curve. *Compositio Mathematica*, 72(2):173–204, 1989.

[9] M. Green and R. Lazarsfeld. Special divisors on curves on a K3 surface. *Inventiones Mathematicae*, 89(2):357–370, 1987.

[10] T. Johnsen and A.L. Knutsen. *K3 Projective Models in Scrolls*. Number no. 1842

[11] A.L. Knutsen. On kth-order embeddings of K3 surfaces and Enriques surfaces. *Manuscripta Mathematica*, 104(2):211–237, 2001.

[12] A.L. Knutsen. On two conjectures for curves on $K3$ surfaces. *Internat. J. Math.*, 20(12):1547–1560, 2009.

[13] I.I. Pyatetskij-Shapiro and I.R. Shafarevich. A Torelli theorem for algebraic surfaces of type K3. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 35:530–572, 1971.

[14] B. Saint-Donat. Projective models of K3 surfaces. *American Journal of Mathematics*, 96(4):pp. 602–639, 1974.

[15] S. Taki. Classification of non-symplectic automorphisms of order 3 on K3 surfaces. *Mathematische Nachrichten*, 284(1):124–135, 2011.

[16] K. Watanabe. The Clifford index of line bundles on a 2-elementary K3 surface given by a double cover of a del Pezzo surface. *Geometriae Dedicata*, pages 1–15, 2014.

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